MATH 2850: INVERSE LAPLACE TRANSFORMS

GOAL: Reverse the process of Laplace Transform.

EXAMPLE: $\mathcal{L}\left\{e^{2t}\right\} = \dfrac{1}{s-2}$, so $\mathcal{L}^{-1}\left\{\dfrac{1}{s-2}\right\} = e^{2t}$

DERIVATION OF FORMULAS:

- $\mathcal{L}\left\{t^n\right\} = \dfrac{n!}{s^{n+1}}$, so $\mathcal{L}^{-1}\left\{\dfrac{1}{s^n}\right\} = \dfrac{t^{n-1}}{(n-1)!}$

- $\mathcal{L}\left\{e^{at}\right\} = \dfrac{1}{s-a}$, so $\mathcal{L}^{-1}\left\{\dfrac{1}{s-a}\right\} = e^{at}$

- $\mathcal{L}\left\{\sin(kt)\right\} = \dfrac{k}{s^2+k^2}$, so $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+k^2}\right\} = \dfrac{1}{k}\sin(kt)$

- $\mathcal{L}\left\{\sinh(kt)\right\} = \dfrac{k}{s^2-k^2}$, so $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2-k^2}\right\} = \dfrac{1}{k}\sinh(kt)$

- $\mathcal{L}\left\{\cos(kt)\right\} = \dfrac{s}{s^2+k^2}$, so $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+k^2}\right\} = \cos(kt)$

- $\mathcal{L}\left\{\cosh(kt)\right\} = \dfrac{s}{s^2-k^2}$, so $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2-k^2}\right\} = \cosh(kt)$

- Linearity: $\mathcal{L}^{-1}\left\{c_1F(s) \pm c_2G(s)\right\} = c_1\mathcal{L}^{-1}\left\{F(s)\right\} \pm c_2\mathcal{L}^{-1}\left\{G(s)\right\} = c_1f(t) \pm c_2g(t)$

- Shift: $\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a)$, so $\mathcal{L}^{-1}\left\{F(s-a)\right\} = e^{at}\mathcal{L}^{-1}\left\{F(s)\right\} = e^{at}f(t)$

  NOTE: Stated differently: $\mathcal{L}^{-1}\left\{F(s)\right\} = e^{at}\mathcal{L}^{-1}\left\{F(s+a)\right\}$

NOTE: The Laplace Transform is defined as an integral ...the Inverse Laplace Transform is also an integral!

The integral for the Inverse Laplace Transform happens in the Complex Plane, however.

EXAMPLE: Find the following inverse Laplace Transforms:

- $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+4}\right\}$

Ans: $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+4}\right\} = \cos(2t)$

- $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2-9}\right\}$

Ans: $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2-9}\right\} = \dfrac{1}{3}\sinh(3t)$

- $\mathcal{L}^{-1}\left\{\dfrac{1}{s+3} - \dfrac{2}{s^4}\right\}$

Ans: $\mathcal{L}^{-1}\left\{\dfrac{1}{s+3} - \dfrac{2}{s^4}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s+3}\right\} - \mathcal{L}^{-1}\left\{\dfrac{2}{s^4}\right\} = e^{-3t} - \dfrac{t^3}{3}$

- $\mathcal{L}^{-1}\left\{\dfrac{3s-1}{s^2+4}\right\}$

Ans: $\mathcal{L}^{-1}\left\{\dfrac{3s-1}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\dfrac{3s}{s^2+4}\right\} - \mathcal{L}^{-1}\left\{\dfrac{1}{s^2+4}\right\} = 3\cos(2t) - \dfrac{1}{2}\sin(2t)$

EXAMPLE: Use Partial Fractions to help you find the following Inverse Laplace Transforms:

- $\mathcal{L}^{-1}\left\{\dfrac{10}{s^2-s-6}\right\}$

Ans: $\mathcal{L}^{-1}\left\{\dfrac{10}{s^2-s-6}\right\}=\mathcal{L}^{-1}\left\{\dfrac{2}{s-3}-\dfrac{2}{s+2}\right\}=\ldots=2e^{3t}-2e^{-2t}$

- $\mathcal{L}^{-1}\left\{\dfrac{s^2+27}{s^3+9s}\right\}$

Ans: $\mathcal{L}^{-1}\left\{\dfrac{s^2+27}{s^3+9s}\right\}=\mathcal{L}^{-1}\left\{\dfrac{3}{s}-\dfrac{2s}{s^2+9}\right\}=\ldots=3-2\cos(3t)$

- Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s^4+4s^2+3}\right\}$

Ans: $\mathcal{L}^{-1}\left\{\dfrac{1}{s^4+4s^2+3}\right\}=\mathcal{L}^{-1}\left\{\dfrac{1}{2}\,\dfrac{1}{s^2+1}-\dfrac{1}{2}\,\dfrac{1}{s^2+3}\right\}=\ldots=\dfrac{1}{2}\sin(t)-\dfrac{1}{2\sqrt{3}}\sin(t\sqrt{3})$

EXAMPLE: Complete the square to help you find the following Inverse Laplace Transforms.

- $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2-4s+13}\right\}$

Ans: $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2-4s+13}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{(s-2)^2+9}\right\} = e^{2t}\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+9}\right\} = ... = \dfrac{1}{3}e^{2t}\sin(3t)$

- $\mathcal{L}^{-1}\left\{\dfrac{2s+3}{s^2-4s+13}\right\}$

Ans: $\mathcal{L}^{-1}\left\{\dfrac{2s+3}{s^2-4s+13}\right\} = \mathcal{L}^{-1}\left\{\dfrac{2s+3}{(s-2)^2+9}\right\} = e^{2t}\mathcal{L}^{-1}\left\{\dfrac{2(s+2)+3}{s^2+9}\right\} = e^{2t}\mathcal{L}^{-1}\left\{\dfrac{2s+7}{s^2+9}\right\} = ...$

$... = 2e^{2t}\cos(3t) + \dfrac{7}{3}e^{2t}\sin(3t)$

HOMEWORK: Section 8.2: Pg. 412: 1 - 8